

Invariants of Quantum Algebras and Their Applications in Physics¹

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Abstract

We summarize recent results^[1] on a functor that maps tangles to commutants of a quasitriangular Hopf algebra. In certain cases, images of $(1, 1)$ -tangles are new Casimir operators of the algebra whose eigenvalues are link invariants. Some applications to physics are briefly discussed.

I. A Functor Mapping Tangles to a Quantum Algebra

1.1 Tangles. An oriented (n, n) -tangle T is the disjoint union of n open oriented strands and an arbitrary number of oriented closed strands embedded in a cylinder in a 3-manifold, with all the n tails of the open strands held fixed on the ceiling of the cylinder and all the n tips held fixed on the floor. A link is a $(0, 0)$ -tangle. A tangle diagram is a regular two-dimensional projection of T . It is composed of positive () and negative () crossings and edges ($\downarrow$). An ambient (regular, resp.) isotopy class of tangle diagrams is defined by the equivalence relations generated by the three (two, resp.) Reidemeister moves^[2], I, II and III (II and III, resp.) on tangle diagrams, generalized for oriented tangle diagrams. Henceforth the adjectives "oriented" and "diagrams" will normally be suppressed.

1.2 Quasitriangular Hopf algebra. A quasitriangular Hopf algebra^[3-5] is a noncommutative, noncocommutative Hopf algebra $\{\mathcal{A}, S, m, \Delta, \epsilon\}$ (resp. algebra over $\mathbb{C}[[\eta]]$, antipode, multiplication, comultiplication, counit) equipped with a universal invertible R -matrix $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$, $\mathcal{R}^{-1} = (S \otimes \text{id})\mathcal{R} = (\text{id} \otimes S^{-1})\mathcal{R}$, satisfying $\forall \alpha \in \mathcal{A} : (i) m(\text{id} \otimes S)\Delta(\alpha) = m(S \otimes \text{id})\Delta(\alpha) = \epsilon(\alpha)1$; (ii) $(T \circ \Delta(\alpha))\mathcal{R} = \mathcal{R}\Delta(\alpha)$; (iii) $(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$, $(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$; (iv) $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$. We call the algebra $\mathcal{U} \supset \{\mathcal{A}, \mathcal{A} \otimes \mathcal{A}, \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}, \dots\}$ a quantum algebra, and define $\mathcal{U}_\Delta \supset \{\mathcal{A}, \Delta\mathcal{A}, (\text{id} \otimes \Delta)\Delta\mathcal{A}, \dots\}$. Let $\mathcal{I}$ be the linear subspace spanned by $ab - ba$, $\forall a, b \in \mathcal{A}$. Let $\mathcal{A}_0$ be the quotient space $\mathcal{A}/\mathcal{I}$, then $\text{Sp}: \mathcal{A} \rightarrow \mathcal{A}_0$ is a linear map such that $\text{Sp}(ab) = \text{Sp}(ba)$, $\forall a, b \in \mathcal{A}$. We consider Sp as a part of the structure of $\mathcal{U}$. For a generic η , $\mathcal{R}$ is sometimes an infinite formal sum in $\mathcal{A} \otimes \mathcal{A}$, which is well defined provided $\mathbb{C}[[\eta]]$ is given an η -adic topology^[6]. In $\mathcal{U}$, there are two invertible elements $v \equiv m((\text{id} \otimes S)\mathcal{R})$ and $v' \equiv m((S \otimes \text{id})T \circ \mathcal{R})$ that satisfy $\forall \alpha \in \mathcal{A} : vav^{-1} = S^{-2}(\alpha)$; $v'av'^{-1} = S^2(\alpha)$; $[vv' = v'v \equiv \lambda, \alpha] = 0^{[1,3]}$.

1.3 The functor $\mathcal{V}$. A functor $\mathcal{V}$ for tangles is a mapping of ambient $[T]$ or regular $[T]_r$ classes of tangles to an algebraic structure. It turns out that, in order to construct $\mathcal{V}$ it is necessary to identify the wrong-way edges in a tangle. For a generic tangle, one way to do this is to examine the Seifert circles in the tangle^[1]; on each of such circles there is one and only one wrong-way edge, denoted by $\Downarrow$ ($\Downarrow$, resp.) if it is on a clockwise (counterclockwise,

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resp.) Seifert circle. It can be shown^[1] that every tangle is regularly isotopic to a partially closed braid, called a braid-tangle. The wrong-way edges on a braid-tangle are just the edges that (partially) close the braid. Henceforth we denote T as a tangle with its wrong-way edges identified and marked as $\vdash$ or $\nless$. We treat the edges and crossings as single or crossed arrows. Let $\alpha, \beta, \dots$ be arrows. The tensor product $\alpha \otimes \beta$ is the disjoint union of the two arrows. The multiplication p adjoins two arrows: $p(\alpha \otimes \beta) = \alpha\beta$. Successive multiplications yield an open string of arrows: $p_{12}(p_{13}(\alpha \otimes \beta \otimes \gamma)) = p_{12}(\alpha\gamma \otimes \beta) = \alpha\gamma\beta$. Given an open string $\tau = \alpha_1\alpha_2 \dots \alpha_k$, denote a closed string $\hat{\tau}_{\{a\}}$ to be the class $\{\alpha_{s_1}\alpha_{s_2} \dots \alpha_{s_k} | (s_1, s_2, \dots, s_k) \in \{\text{all cyclic perm. of } (1, 2, \dots, k)\}\}$. We define a functor $\mathcal{V}$ whose actions on the arrows in T are: $\mathcal{V}(\downarrow) = 1$; $\mathcal{V}(\nless) = v$; $\mathcal{V}(\vdash) = v'$; $\mathcal{V}(\nless) = R$; $\mathcal{V}(\nless) = R'$; $\mathcal{V}(\alpha \otimes \beta) = \mathcal{V}(\alpha) \otimes \mathcal{V}(\beta)$; $\mathcal{V}(\alpha\beta) = \mathcal{V}(\alpha)\mathcal{V}(\beta) = m(\mathcal{V}(\alpha) \otimes \mathcal{V}(\beta))$; $\mathcal{V}(\hat{\tau}) = \text{Sp}(\mathcal{V}(\tau_*)) \in \mathcal{A}_0$, $\tau_* \in \hat{\tau}$. Let $\{T\}$ be the set of all tangles and $[T_l^{(n)}]_r$ be a regular isotopy of (n, n) -tangles with l closed strands. Then $\mathcal{V}$ is a functor for regular isotopic tangles^[1]:

$$\mathcal{V} : \{T\} \rightarrow \mathcal{U}, \quad \mapsto \mathcal{V}[T_l^{(n)}]_r = \mathcal{V}(T) \in \mathcal{A}^{\otimes n} \otimes \mathcal{A}_0^{\otimes l}, \quad \forall T \in [T_l^{(n)}]_r. \quad (1)$$

1.4 Representation of $\mathcal{V}$. Let V be the vector space supporting $\mathcal{A}$ and let $\pi \in \text{End}(V)$ be an irreducible representation of $\mathcal{A}$. Then $\pi(\lambda) = \lambda_\pi \mathbf{1}_\pi$. Suppose $\lambda_\pi \neq 0$. Define $h_\pi \equiv \lambda_\pi^{-1/2} \pi(v)$, $h'_\pi \equiv \lambda_\pi^{-1/2} \pi(v')$, $\mathcal{R}_\pi \equiv (\pi \otimes \pi)\mathcal{R}$, etc. Let Tr_π be the matrix trace in π . Then $\mathcal{V}'_\pi$ defined as $\mathcal{V}$ with the set $\{\mathcal{R}, \mathcal{R}', v, v', \text{Sp}\}$ substituted by $\{\mathcal{R}_\pi, \mathcal{R}'_\pi, h_\pi, h'_\pi, \text{Tr}_\pi\}$ is obviously a functor that sends $[T_l^{(n)}]_r$ to $(\pi(\mathcal{A}))^{\otimes n}$. Note that $\text{Sp} : \mathcal{A} \rightarrow \mathcal{A}_0$ but $\text{Tr}_\pi : \mathcal{A} \rightarrow \mathbb{C}$. Furthermore, if $w(T)$ is the writhe number of a tangle T , then

$$\mathcal{V}_\pi[T] \equiv \lambda_\pi^{w(T)/2} \mathcal{V}'_\pi(T) \quad (2)$$

is an invariant for ambient isotopic tangles^[1].

II. Commutants and Tangle Casimir Operators of Quantum Algebras

2.1 The quantum algebra $\tilde{\mathcal{U}}$. Consider a more restrictive type of $\mathcal{U}$ denoted by $\tilde{\mathcal{U}}$, which is a quantum algebra with properties: (i) Generated in the Chevalley basis by $\{H_i, X_i^+, X_i^-; i = 1, \dots\}$, with the H_i 's generating the Cartan subalgebra; (ii) $\Delta H_i = H_i \otimes 1 + 1 \otimes H_i$, $\Delta X_i^\pm = X_i^\pm \otimes q^{H_i/2} + q^{-H_i/2} \otimes X_i^\pm$. Then $S(H_i) = -H_i$, $S(X_i^\pm) = -q^{H_i/2} X_i^\pm q^{-H_i/2}$. Examples of $\tilde{\mathcal{U}}$ include $\mathcal{U}_q(g)$ over $\mathbb{C}[[\eta]]$, $q \equiv e^\eta$, where g is a simple complex Lie algebra, or is the graded $\mathfrak{sl}(M|N)$.

2.2 Every $\mathcal{V}[T]_r$ almost commutes with $\tilde{\mathcal{U}}_\Delta$. Let $\sigma : T_l^{(n)} \rightarrow S_n$ be the natural mapping of tangle diagrams to the symmetry groups. The map does not depend on the number l of the closed strands in $T_l^{(n)}$. Let $\Delta^{\{1\}} = \Delta$ and define, for $n \geq 2$, $\Delta^{\{n\}} = (\text{id} \otimes \text{id} \dots \otimes \Delta) \Delta^{\{n-1\}} = (\text{id}^{\otimes(n-1)} \otimes \Delta) \Delta^{\{n-1\}}$. By definition, $\mathcal{A}^{\{n\}} \equiv \Delta^{\{n-1\}} \mathcal{A} \in \tilde{\mathcal{U}}_\Delta \subset \tilde{\mathcal{U}}$. Then^[1], for any $T_l^{(n)}$,

$$(\Delta^{\{n-1\}}(\alpha) \otimes 1^{\otimes l}) \mathcal{V}[T]_r = \mathcal{V}[T]_r (\sigma(T) \circ \Delta^{\{n-1\}}(\alpha)) \otimes 1^{\otimes l}, \quad \forall \alpha \in \mathcal{A}. \quad (3)$$

This generalizes the coboundary condition $(T \circ \Delta(\alpha))\mathcal{R} = \mathcal{R}\Delta(\alpha)$.

2.3 Commutants of $\tilde{\mathcal{U}}_\Delta$. Let $\mathcal{P} : [T_l^{(n)}] \rightarrow S_n$ with $\mathcal{P}[T_l^{(n)}]$ acting on $V^{\otimes n}$. Then^[1]

$$[(T_n \circ \mathcal{V}[T_l^{(n)}]_r) \mathcal{P}^{-1}(T_l^{(n)}), \Delta^{\{n-1\}}(\alpha) \otimes 1^{\otimes l}] = 0, \quad \forall \alpha \in \mathcal{A}. \quad (4)$$

A well-known special case for $n = 1, l = 0$ is $\mathcal{P}(\mathcal{R})\mathcal{R}$, which commutes with $\Delta\mathcal{A}$.

2.4 Tangle Casimir operators of $\tilde{\mathcal{U}}_\Delta$. For T being a $(1, 1)$ -tangle with l closed strands, we call $\mathcal{V}[T]_r$ a tangle Casimir operator of $\tilde{\mathcal{U}}_\Delta$; it commutes with $\mathcal{A} \otimes 1^{\otimes l}$. As an example, let $\mathcal{R} = \sum_i a_i \otimes b_i$. Then for T_{trefoil} the tangle that closes to a trefoil, $\mathcal{V}[T_{\text{trefoil}}]_r =$

$\sum_{s,t,r} a_s b_t a_r v b_s a_t b_r$ commutes with $\mathcal{A}$. Generally, it follows from Eq. (3) and Schur's lemma that in an irreducible representation π of $\tilde{U}$, for T being a $(1,1)$ -tangle with any number of closed strands,

$$\mathcal{V}_\pi[T] = Q_\pi[T] \mathbf{1}_\pi, \quad (5)$$

where $Q_\pi[T]$ is a $\mathbb{C}$ -valued eigenvalue.

2.5 Link invariants are eigenvalues of tangle Casimir operators. Let T be a $(1,1)$ -tangle and $\hat{T}$ its closure, which is a link. Then $Q_\pi[T]$ is a link invariant for $[\hat{T}]$ if (a) $\text{Tr}_\pi(h_\pi) \neq 0$, or (b) the multiplicities in the decomposition of $\pi \otimes \pi$ are not greater than one and $\mathcal{V}_\pi$ is sufficiently nontrivial^[1]. Condition (a) is met by all the representations of $U_q(g)$ with g being a simple complex Lie algebra as well as $g = \mathfrak{sl}(n|m)$, $n \neq m$, for generic q . It is not met by $U_q(\mathfrak{sl}(m|m))$. Condition (b) is met by all the fundamental representations of the quantum algebras mentioned above, for arbitrary q .

III. Some Applications in Physics

3.1 Conserved topological charges in 2d QFT. Recently it has been shown^[7,8] that some integrable quantum field theories in two dimensions have explicit quantum algebra symmetries. Such systems should have an infinite number of conserved quantities, called topological charges, one for each distinct tangle Casimir operator. In representations of the quantum algebra that satisfy the conditions in subsection 2.5, these conserved topological charges should be link invariants.

3.2 The Alexander-Conway polynomial. For the fundamental 2×2 representation of $U_q(\mathfrak{sl}(1|1))$, $\text{Tr}_\pi(h_\pi) = 0$ so $\mathcal{V}_\pi[L] = 0$ for all the links. But Q_π satisfies^[9,10] the skein relation, $Q_\pi[\text{crossing}] + (q - q^{-1})Q_\pi[\downarrow\downarrow] - Q_\pi[\text{crossing}] = 0$, and therefore gives the Alexander-Conway polynomial^[11].

3.3 Quantum holonomy and Chern-Simons theory in three dimensions. Recently^[12] it was shown that in a quantum gauge theory, the group-valued quantum holonomy, $\Phi(C_x) = \langle P \exp(i \oint_{C_x} A dx) \rangle$, where C_x is a closed contour with the initial point $x \in C$, is gauge invariant and x -independent. This implied that for the Chern-Simons theory in three dimensions, $\Phi(L_x)$ is a link invariant. Earlier Witten^[13] has already shown that the Wilson loop, or character-valued $W_\pi = \text{Tr}_\pi(\Phi[L])$ is a link invariant. Thinking of L_x as a $(1,1)$ -tangle, the pair $\{\Phi[L_x], W_\pi[L]\}$ to 3d CST is what $\{\mathcal{V}[L_x], \mathcal{V}_\pi[L]\}$ of subsection 2.4 is to a quantum algebra. If in an irreducible π , $\pi(\Phi[L_x]) = P_\pi[L] \mathbf{1}_\pi$, then $W_\pi[L] = P_\pi(L) \text{Tr}_\pi(\mathbf{1}_\pi)$. An application of this result is that for the gauge group $\text{SL}(M|M)$, $W_\pi[L] \equiv 0$ (because Tr is replaced by the supertrace) while $P_\pi[L] \neq 0$ is the Alexander-Conway polynomial^[12].

3.4 Quantum spin chains. For a $(2,2)$ -tangle $T^{(2)}$, $\tilde{\mathcal{H}} \equiv \mathcal{P}(T^{(2)}) \mathcal{V}_\pi(T^{(2)})$ commutes with $\pi \otimes \pi(\Delta \mathcal{A})$. Consider a tensored space of length N and let $\tilde{\mathcal{H}}_{i,i+1}$ act on the i -th and $(i+1)$ -th spaces. Then the Hamiltonian for an open quantum spin chain with nearest neighbor interaction, $H = \sum_{i=1}^{N-1} \tilde{\mathcal{H}}_{i,i+1}$, is $\tilde{U}_\Delta$ invariant and satisfies the first requirement for H to be exactly soluble^[14]. (Exact solubility^[15,16] also requires that $\tilde{\mathcal{H}}$ satisfies the Yang-Baxter equation^[17,18].) More generally, for any (n,n) -tangle T , define $\tilde{\mathcal{H}}^{(n)} \equiv (T_n \circ \mathcal{V}_\pi[T]) \mathcal{P}^{-1}(T)$, then the Hamiltonian for an open quantum spin chain with $(n-1)$ -th nearest neighbor interactions, $H^{(n)} = \sum_{i=1}^{N-n+1} \tilde{\mathcal{H}}_{i,i+1,\dots,i+n-1}^{(n)}$, is $\tilde{U}_\Delta$ invariant. As well, being itself $\tilde{U}_\Delta$ invariant, $\tilde{\mathcal{H}}^{(n)}$ also satisfies one of the basic requirements for being a transfer matrix.

For π being the fundamental 2×2 representation of $U_q(\mathfrak{sl}(2))$, $\pi \otimes \pi = \underline{3} \oplus \underline{1}$, and $\tilde{\mathcal{H}}$ has two eigenvalues $\epsilon_{\underline{3}}$ and $\epsilon_{\underline{1}}$. Expressed in terms of Pauli matrices,

$$H_{\text{sl}(2)} = \frac{1}{q^{-1} + q} \left[(\epsilon_{\underline{3}} - \epsilon_{\underline{1}}) H_{PS} + \frac{q^{-1} + q}{4} (3\epsilon_{\underline{3}} + \epsilon_{\underline{1}}) \right], \quad (6)$$

$$H_{PS} = \sum_{i=1}^{N-1} \left[\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \frac{q^{-1} + q}{4} \sigma_i^3 \sigma_{i+1}^3 + \frac{q^{-1} - q}{4} (\sigma_i^3 - \sigma_{i+1}^3) \right] \quad (7)$$

is the soluble Hamiltonian of Pasquier and Saleur^[14] for an open spin chain. The infinite classes of $(2, 2)$ -tangles have, in this case, projected onto two degrees of freedom that affect $H_{\text{sl}(2)}$ trivially: (i) an global scale factor; and (ii) a constant term. In particular, H_{PS} has no dependence on the eigenvalues. Hence the Hamiltonian derived from any $(2, 2)$ -tangle (whose image in $\mathcal{U}_q(\text{sl}(2))$ has two nondegenerate eigenvalues) is soluble; changing the Hamiltonian associated with one tangle to a Hamiltonian associated with another tangle amounts to renormalizing the Hamiltonian in such a way that the structure of the energy spectrum is unchanged.

The situation for $\mathcal{U}_q(\text{sl}(1|1))$ is similar: $\pi \otimes \pi = \underline{2} \oplus \underline{2}'$ and,

$$H_{\text{sl}(1|1)} = \frac{1}{q^{-1} + q} \left[(\epsilon_2 - \epsilon'_2) H_{FW} + \frac{q^{-1} + q}{2} (\epsilon_2 + \epsilon'_2) \right], \quad (8)$$

where

$$H_{FW} = \sum_{i=1}^{N-1} \left[\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \frac{1}{2} (q^{-1} \sigma_i^3 + q \sigma_{i+1}^3) \right] \quad (9)$$

is the Hamiltonian for free-fermions^[19]. We see that the $\mathcal{U}_q(\text{sl}(2))$ and $\mathcal{U}_q(\text{sl}(1|1))$ symmetries for open quantum spin chains are extremely robust. This probably will not be the case for higher representations of these quantum algebras or for larger quantum algebras.

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